有限元是如何求解边值问题的？

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从弹性力学案例出发

➢一维应力问题

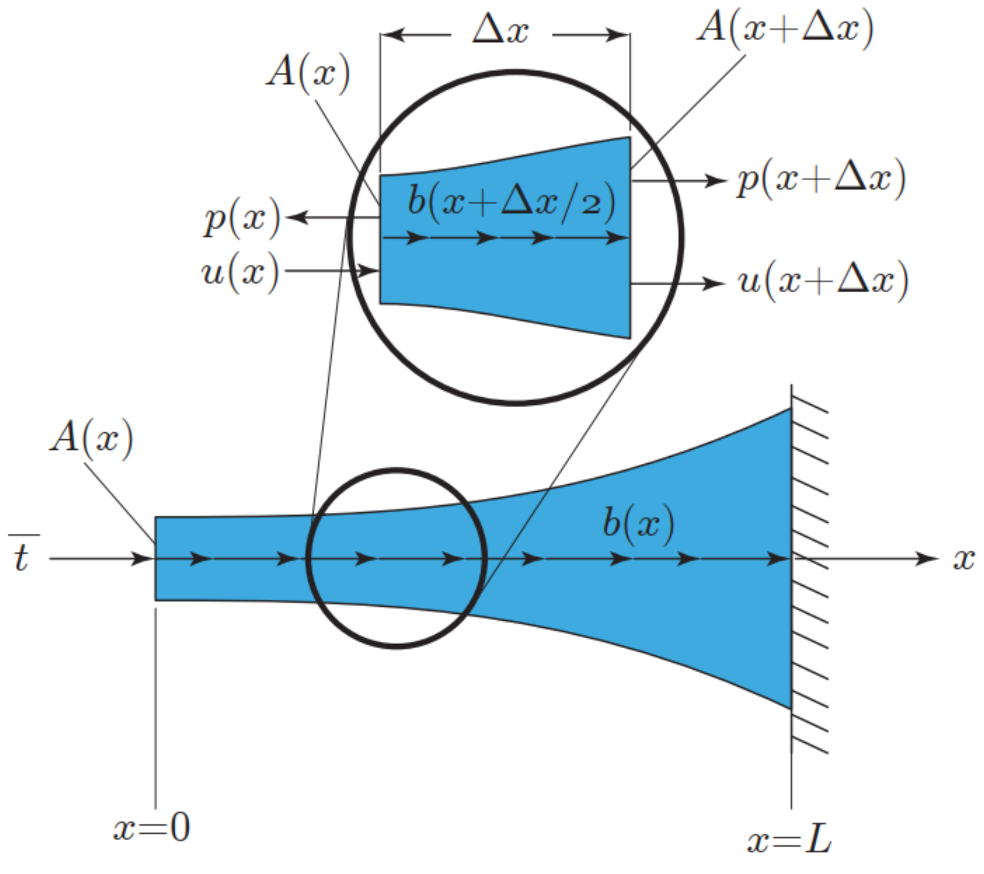

Figure 2.1: A one-dimensional stress analysis (elasticity) problem

- 微元平衡方程

$$-p(x)+b\left(x+\frac{\Delta x}{2}\right)\Delta x+p(x+\Delta x)=0$$

$$\Rightarrow \frac{p(x+\Delta x)-p(x)}{\Delta x}+b\left(x+\frac{\Delta x}{2}\right)=0$$

$$\Rightarrow \frac{dp(x)}{dx}+b(x)=0$$

$$p(x)=\sigma(x)A(x) \qquad \sigma(x)=E(x)\varepsilon(x)$$

$$\varepsilon(x)=\frac{du(x)}{dx}$$

$$\boxed{\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right)+b(x)=0 \qquad 0<x<L}$$

边值问题

➢一维应力问题

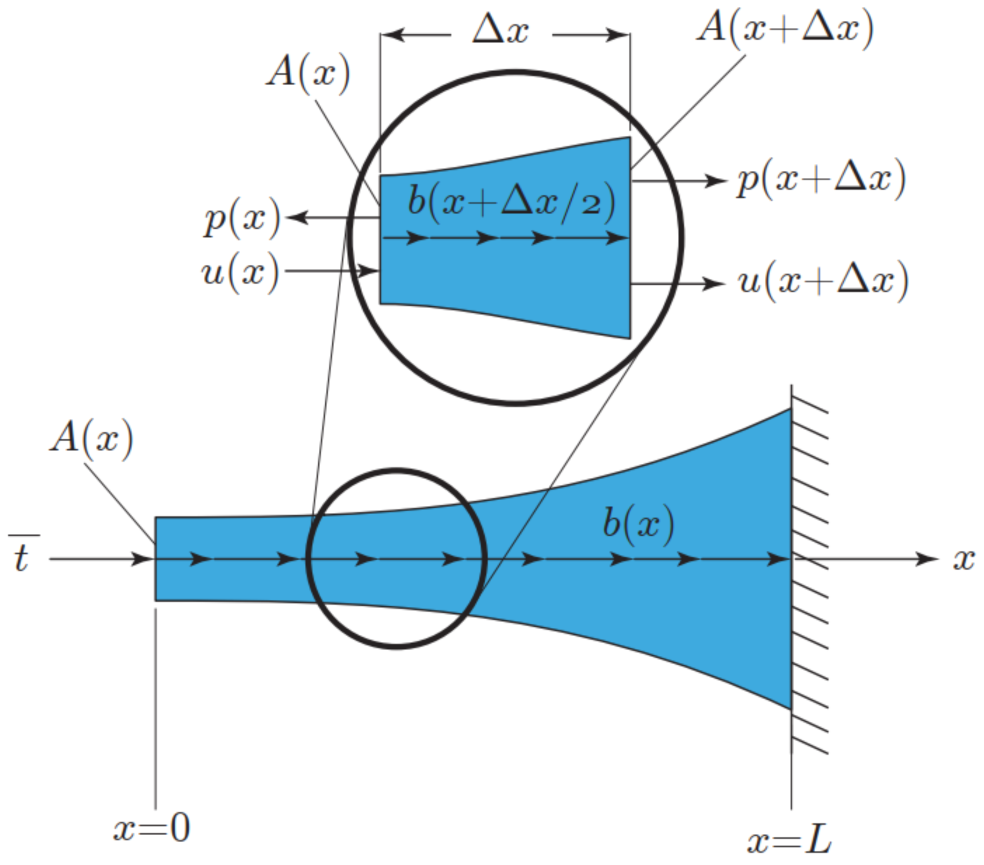

Figure 2.1: A one-dimensional stress analysis (elasticity) problem

$$\sigma = E\epsilon$$

- 控制方程（二阶微分）

$$\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) + b(x) = 0 \quad , \quad 0 < x < L$$

- 边界条件

$$u(x = L) = 0$$

$$u'(x = 0) = -\frac{\bar{t}}{E(x = 0)}$$

微分方程+边界条件 ⟹ 边值问题

Boundary value problem

有限元基本方法

➢有限元方法求解边值问题：一维问题示例

微分形式为强形式

$$-u''(x) = f(x)\,,\quad 0 < x < 1\,,\quad u(0) = 0\,,\quad u(1) = 0\,,$$

- Step-1: 构建弱形式，或变分形式。等式两边同乘测试函数 $v(x)$，

  满足$v(0) = 0$，$v(1) = 0$，

$$-u''v = fv\,,$$

  然后从0到1积分，并用分部积分法

$$\int_0^1 (-u''v)\,dx = \; -u'v\Big|_0^1 + \int_0^1 u'v'\,dx$$

$$= \int_0^1 u'v'\,dx$$

积分形式为弱形式

$$\Longrightarrow \quad \int_0^1 u'v'\,dx = \int_0^1 fv\,dx\,,\ \text{the weak form.}$$

有限元基本方法

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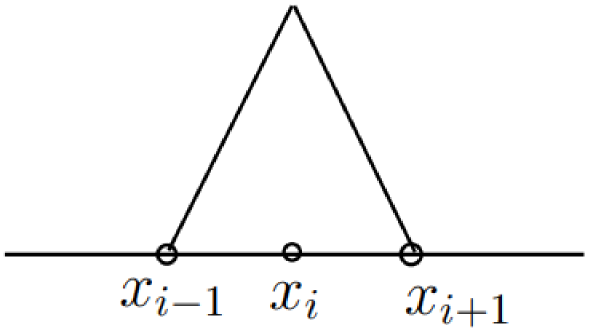
- Step-2: 离散、划分网格。按坐标轴均匀划分为n份。

$$x_i = i\,h,\ i = 0, 1, \cdots, n, \quad \text{where } h = 1/n$$

$$(x_{i-1}, x_i),\ i = 1, 2, \cdots, n.$$

- Step-3: 基于网格构建基函数，如分段线性函数。

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{h} & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases}$$



$$(i = 1, 2, \cdots, n-1)$$

有限元基本方法

- Step-4: 通过基函数的线性组合构建近似有限元解。

$$u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x)\,,$$

c_j 为待确定参数。

$u_h(x)$也是分段线性函数，是一个近似解。

$$\int_0^1 u_h' v' dx = \int_0^1 f v dx\,,$$

数值计算误差由此“引入”

$$\Longrightarrow \quad \int_0^1 \sum_{j=1}^{n-1} c_j \phi_j' v' dx = \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j' v' dx = \int_0^1 f v\, dx\,.$$

有限元基本方法

$$\Longrightarrow \int_0^1 \sum_{j=1}^{n-1} c_j \phi_j' v' \, dx = \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j' v' \, dx = \int_0^1 f v \, dx \, .$$

依次将v(x)选择为 $\phi_1, \phi_2, \cdots, \phi_{n-1}$ 数值计算误差再次被“引入”

$$\left(\int_0^1 \phi_1'\phi_1' dx\right) c_1 + \cdots + \left(\int_0^1 \phi_1'\phi_{n-1}' dx\right) c_{n-1} = \int_0^1 f\phi_1 dx$$

$$\left(\int_0^1 \phi_2'\phi_1' dx\right) c_1 + \cdots + \left(\int_0^1 \phi_2'\phi_{n-1}' dx\right) c_{n-1} = \int_0^1 f\phi_2 dx$$

$$\cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\left(\int_0^1 \phi_i'\phi_1' dx\right) c_1 + \cdots + \left(\int_0^1 \phi_i'\phi_{n-1}' dx\right) c_{n-1} = \int_0^1 f\phi_i dx$$

$$\cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\left(\int_0^1 \phi_{n-1}'\phi_1' dx\right) c_1 + \cdots + \left(\int_0^1 \phi_{n-1}'\phi_{n-1}' dx\right) c_{n-1} = \int_0^1 f\phi_{n-1} dx \, ,$$

有限元基本方法

写成矩阵形式：

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & \cdots & a(\phi_1,\phi_{n-1}) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & \cdots & a(\phi_2,\phi_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ a(\phi_{n-1},\phi_1) & a(\phi_{n-1},\phi_2) & \cdots & a(\phi_{n-1},\phi_{n-1}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_{n-1}) \end{bmatrix},$$

where

$$a(\phi_i,\phi_j) = \int_0^1 \phi_i'\phi_j' dx\,, \qquad (f,\phi_i) = \int_0^1 f\phi_i dx\,.$$

有限元基本方法

<u>注意：</u> $\phi_i(x)$:

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{h} & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases}$$

待求解系数

$$\begin{bmatrix} \frac{2}{h} & -\frac{1}{h} & & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \int_0^1 f\phi_1 dx \\ \int_0^1 f\phi_2 dx \\ \int_0^1 f\phi_3 dx \\ \vdots \\ \int_0^1 f\phi_{n-2} dx \\ \int_0^1 f\phi_{n-1} dx \end{bmatrix}$$

有限元基本方法

- Step-5: 求解线性方程组。获得系数 c_j

得到近似解 $u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x)\,,$

- Step-6: 误差分析。

$$-u''(x) = f(x)\,, \quad 0 < x < 1\,, \quad u(0) = 0\,, \quad u(1) = 0\,,$$

当 $f(x) = 1\,,$ 解析解为 $u(x) = \dfrac{x(1-x)}{2}\,.$