# Principle of Minimum Potential Energy, Variational Analysis, and Finite element method: An Elastic Bar Under Tension as example

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## 1 Introduction

The *Principle of Minimum Potential Energy* is a fundamental concept in structural mechanics, stating that an elastic system in equilibrium adopts a configuration that minimizes its total potential energy. This principle is the foundation of variational methods used in elasticity to derive the equations governing the behavior of structures under load.

## 2 Problem Setup

Consider a one-dimensional elastic bar with the following properties:

- Length: $L$
- Cross-sectional area: $A$
- Young's modulus: $E$
- Fixed at $x = 0$, free at $x = L$
- Subjected to an external force $F$ at the free end $x = L$

We aim to determine the equilibrium displacement $u(x)$ of the bar using variational principles.

## 3 Principle of Minimum Potential Energy

### 3.1 Potential Energy

The total potential energy $\Pi$ of the system is the sum of the strain energy $U$ stored in the bar and the work $W$ done by external forces:

$$\Pi = U - W, \tag{1}$$

where:

- $U$ is the strain energy, which represents the elastic energy stored in the bar due to deformation,
- $W$ is the work done by external forces.

The strain energy $U$ represents the energy stored in the bar due to deformation. For a linear elastic material, the total strain energy can be expressed as:

$$U = \int_V \frac{1}{2}\sigma\varepsilon \, dV. \tag{2}$$

For a one-dimensional elastic bar with a uniform cross-sectional area $A$, the volume element is $dV = Adx$, so the strain energy simplifies to:

$$U = \int_0^L \frac{1}{2}\sigma\varepsilon A\,dx, \tag{3}$$

where:

- $\sigma$ is the axial stress in the bar,
- $\varepsilon$ is the axial strain,
- $A$ is the cross-sectional area,
- $L$ is the length of the bar.

For a linear elastic material, the stress-strain relationship follows Hooke's law:

$$\sigma = E\varepsilon. \tag{4}$$

The strain $\varepsilon$ is the derivative of the displacement:

$$\varepsilon = \frac{du}{dx}. \tag{5}$$

Substituting this into the strain energy expression:

$$U = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx. \tag{6}$$

The work $W$ done by the external force $F$ at the free end of the bar is:

$$W = Fu(L), \tag{7}$$

where $u(L)$ is the displacement at the free end of the bar.

## 3.2 First Variation of Total Potential Energy

To find the equilibrium displacement, we set the first variation of the total potential energy $\delta\Pi$ to zero. This leads to the equilibrium condition:

$$\delta\Pi = \delta U - \delta W = 0. \tag{8}$$

### 3.2.1 First Variation of Strain Energy $U$

Here we give two methods to calculate the first variation of strain energy.

**Method 1: Explicitly Calculating $\delta U = U(u + \delta u) - U(u)$**

Introduce a small variation $\delta u(x)$ in the displacement field. The strain energy for the perturbed displacement is then:

$$U(u + \delta u) = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx} + \frac{d(\delta u)}{dx}\right)^2 dx. \tag{9}$$

The variation of the strain energy, $\delta U$, is defined as the difference between the strain energy corresponding to the perturbed displacement and the original strain energy:

$$\delta U = U(u + \delta u) - U(u). \tag{10}$$

Substituting the expressions for $U(u + \delta u)$ and $U(u)$ gives:

$$\delta U = \int_0^L \frac{1}{2}EA\left[\left(\frac{du}{dx} + \frac{d(\delta u)}{dx}\right)^2 - \left(\frac{du}{dx}\right)^2\right] dx. \tag{11}$$

Since $\delta u(x)$ is an infinitesimally small variation, the quadratic term $\left(\frac{d(\delta u)}{dx}\right)^2$ is of second order and can be neglected. Thus, the expression simplifies to:

$$\delta U = \int_0^L EA\frac{du}{dx}\frac{d(\delta u)}{dx}\,dx. \tag{12}$$

**Method 2: Using $\int \sigma \, \delta\varepsilon \, dV$**

The first variation of strain energy can also be calculated directly using the stress $\sigma$ and the variation of strain $\delta\varepsilon$, without explicitly expanding $U(u+\delta u) - U(u)$. This approach leverages the constitutive relationship and is more efficient for variational analysis.

For a linear elastic material, the variation of strain energy is:

$$\delta U = \int_V \sigma \, \delta\varepsilon \, dV, \tag{13}$$

where $\delta\varepsilon$ is the variation of strain caused by a virtual displacement $\delta u$.

Recall that

$$\sigma = EA\frac{du}{dx}, \quad \varepsilon = \frac{du}{dx}, \quad \text{and} \quad \delta\varepsilon = \frac{d(\delta u)}{dx}. \tag{14}$$

Substitute these into $\delta U$:

$$\delta U = \int_0^L EA\frac{du}{dx}\frac{d(\delta u)}{dx}\, dx. \tag{15}$$

This result matches the expression derived in Method 1 as shown in Eq. 12, confirming that:

$$\delta U = \int_0^L EA\frac{du}{dx}\frac{d(\delta u)}{dx}\, dx. \tag{16}$$

Applying integration by parts:

$$\delta U = \left[EA\frac{du}{dx}\delta u\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u \, dx. \tag{17}$$

### 3.2.2 First Variation of External Work $W$

The variation of external work is defined as the difference between the external work corresponding to the perturbed displacement and the original external work:

$$\delta W = W(u+\delta u) - W(u). \tag{18}$$

For a concentrated force $F$ applied at $x = L$, the variation of external work simplifies to:

$$\delta W = F\delta u(L). \tag{19}$$

## 3.3 Stationary Condition

Setting $\delta\Pi = 0$ leads to the stationary condition:

$$\left[EA\frac{du}{dx}\delta u\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u \, dx - F\delta u(L) = 0. \tag{20}$$

Since $\delta u(x)$ is an arbitrary variation (except at the fixed end $x = 0$, where $\delta u(0) = 0$), the integrand must vanish for the equation to hold for all possible $\delta u(x)$. This gives the governing differential equation:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0. \tag{21}$$

Additionally, the boundary term at $x = L$ must satisfy:

$$EA\frac{du}{dx}\bigg|_{x=L} = F. \tag{22}$$

This is the natural boundary condition at the free end of the bar, representing the equilibrium of forces.

### 3.4 Solution for Displacement

Integrating the governing equation $\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$, we obtain:

$$EA\frac{du}{dx} = C, \tag{23}$$

where $C$ is a constant. Thus, the displacement gradient is constant.

Using the boundary condition at $x = L$:

$$EA\frac{du}{dx}\bigg|_{x=L} = F. \tag{24}$$

$$\frac{du}{dx} = \frac{F}{EA}. \tag{25}$$

Integrating this gives the displacement as a function of $x$:

$$u(x) = \frac{F}{EA}x. \tag{26}$$

This represents a linear displacement profile, indicating that the bar undergoes uniform stretching under tension.

## 4 Finite Element Method (FEM) Formulation

### 4.1 Problem Formulation

Consider a more complex case where the bar has a variable cross-section along the x-axis, meaning the cross-sectional area $A$ is a function of $x$, denoted as $A(x)$. The governing equation and boundary conditions for this problem are given by:

$$\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) = 0, \quad 0 < x < L \tag{27}$$

$$u(0) = 0 \tag{28}$$

$$EA(L)\left.\frac{du}{dx}\right|_{x=L} = F \tag{29}$$

This boundary value problem is generally difficult to solve analytically. Instead, the finite element method (FEM) provides a numerical approach to approximate its solution.

### 4.2 Finite Element Discretization

The bar is divided into $N$ elements of equal length $L_e = L/N$. Each element consists of two nodes located at its endpoints, with nodal displacements denoted as $u_1$ (left node) and $u_2$ (right node). Within each element, the cross-sectional area is assumed to be constant. The displacement field $u(x)$ is approximated using linear shape functions, resulting in a piecewise linear interpolation over the entire domain.

### 4.3 Shape Functions and Displacement Approximation

For a typical element, we introduce a local coordinate $\xi \in [0, 1]$, where $\xi = x/L_e$. The displacement field within the element is approximated using linear shape functions, defined as:

$$N_1(\xi) = 1 - \xi, \quad N_2(\xi) = \xi. \tag{30}$$

These shape functions satisfy the following properties:

- $N_1(0) = 1$, $N_1(1) = 0$; $N_2(0) = 0$, $N_2(1) = 1$ (Kronecker delta property).

- Partition of unity: $N_1(\xi) + N_2(\xi) = 1$.

Using these shape functions, the displacement field within an element can be expressed as:

$$u^e(x) = N_1(\xi)u_1 + N_2(\xi)u_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{31}$$

## 4.4 Element Stiffness Matrix Derivation

### 4.4.1 Strain-Displacement Relationship

Considering small-strain assumption, the axial strain is obtained by differentiating the displacement field:

$$\begin{aligned} \varepsilon &= \frac{du^e}{dx} = \frac{1}{L_e}\frac{du^e}{d\xi} \\ &= \frac{1}{L_e}\left(\frac{dN_1}{d\xi}u_1 + \frac{dN_2}{d\xi}u_2\right) \\ &= \frac{1}{L_e}(-u_1 + u_2) \end{aligned} \tag{32}$$

In matrix form, this becomes:

$$\varepsilon = \mathbf{B}\mathbf{u}^e, \quad \text{where } \mathbf{B} = \frac{1}{L_e}\begin{bmatrix} -1 & 1 \end{bmatrix}, \quad \mathbf{u}^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{33}$$

### 4.4.2 Strain Energy Formulation

The element strain energy is expressed as:

$$U^e = \frac{1}{2}\int_0^{L_e} \sigma\varepsilon A dx = \frac{1}{2}\int_0^{L_e} EA\varepsilon^2 dx \tag{34}$$

Substituting the strain-displacement relationship:

$$U^e = \frac{1}{2}(\mathbf{u}^e)^T \underbrace{\left(\int_0^{L_e} EA\mathbf{B}^T\mathbf{B}dx\right)}_{\mathbf{k}^e} \mathbf{u}^e \tag{35}$$

### 4.4.3 Matrix Integration and Assembly

Evaluating the integral for constant material properties:

$$\begin{aligned} \mathbf{k}^e &= EA\int_0^{L_e} \mathbf{B}^T\mathbf{B}dx \\ &= \frac{EA}{L_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_0^{L_e} dx \\ &= \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{36}$$

### 4.4.4 Physical Interpretation of Stiffness Matrix

The resulting element stiffness matrix:

$$\mathbf{k}^e = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{37}$$

has distinct mechanical interpretations:

- Diagonal terms: Resistance to direct nodal displacements
- Off-diagonal terms: Coupling between nodal displacements
- Symmetry: Satisfies Newton's third law ($k_{12} = k_{21}$)
- Singularity: Requires boundary constraints for solvability

## 4.5 Global Stiffness Matrix Assembly

Consider a bar composed of three elements arranged sequentially from left to right, where each element has a cross-sectional area $A_1$, $A_2$, and $A_3$, respectively. Our goal is to assemble the element stiffness matrices into a global stiffness matrix.

### 4.5.1 Element Stiffness Matrices

Each element $e$ has a local stiffness matrix given by:

$$\mathbf{k}^e = \frac{EA_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{38}$$

where $A_e$ is the cross-sectional area of the element, $L_e$ is its length, and $E$ is Young's modulus.

For the three elements, the local stiffness matrices are:

$$\mathbf{k}^1 = \frac{EA_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{k}^2 = \frac{EA_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{k}^3 = \frac{EA_3}{L_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{39}$$

### 4.5.2 Global Stiffness Matrix Formation

The global stiffness matrix is obtained by assembling the individual element stiffness matrices while ensuring compatibility at shared nodes. The global displacement vector consists of nodal displacements $u_1, u_2, u_3$, and $u_4$, corresponding to the four nodes in the system.

Each element contributes to the global stiffness matrix at the corresponding nodes:

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} + k_{33} & k_{23} & 0 \\ 0 & k_{32} & k_{33} + k_{44} & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{bmatrix} \tag{40}$$

where the individual terms are computed as:

$$\begin{aligned} k_{11} &= \frac{EA_1}{L_1}, \quad k_{12} = k_{21} = -\frac{EA_1}{L_1}, \\ k_{22} &= \frac{EA_1}{L_1} + \frac{EA_2}{L_2}, \quad k_{23} = k_{32} = -\frac{EA_2}{L_2}, \\ k_{33} &= \frac{EA_2}{L_2} + \frac{EA_3}{L_3}, \quad k_{34} = k_{43} = -\frac{EA_3}{L_3}, \\ k_{44} &= \frac{EA_3}{L_3}. \end{aligned}$$

## 4.6 Applying Boundary Conditions

Since the bar is fixed at $x = 0$, we enforce the boundary condition:

$$u_1 = 0. \tag{41}$$

This modifies the system by eliminating the first row and column, reducing the system to three unknown displacements $u_2, u_3$, and $u_4$. The reduced global stiffness matrix is solved along with the applied forces to determine the nodal displacements.

Since the bar is fixed at $x = 0$ and loaded at the right end with a tensile force $F$, the global force vector $\mathbf{F}$ consists of nodal forces at each node. Given that external forces act only at the last node (node 4), the force vector takes the form:

$$\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F \end{bmatrix}. \tag{42}$$

Applying the boundary condition $u_1 = 0$, we remove the first row and column of the stiffness matrix, leading to the reduced system:

$$\mathbf{K}_{\text{reduced}}\mathbf{u} = \begin{bmatrix} k_{22} & k_{23} & 0 \\ k_{32} & k_{33} & k_{34} \\ 0 & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F \end{bmatrix}. \tag{43}$$

This system of equations can now be solved numerically to obtain the nodal displacements $u_2, u_3$, and $u_4$. Once these displacements are known, strain and stress distributions along the bar can be computed.

# 5 Nonlinear Analysis Considerations

## 5.1 Geometric Stiffness and Green-Lagrange Strain

In problems involving large displacements but small strains, the axial force introduces additional geometric stiffness that must be considered. These nonlinear effects are captured using the Green-Lagrange strain, which accounts for the deformation of the material under large displacements.

The Green-Lagrange strain $\varepsilon_{\text{GL}}$ in 1D is defined as:

$$\varepsilon_{\text{GL}} = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 = \varepsilon_L + \varepsilon_{NL}, \tag{44}$$

where:

- $\varepsilon_L = \frac{du}{dx}$: Linear strain, which assumes small deformations,
- $\varepsilon_{NL} = \frac{1}{2}\left(\frac{du}{dx}\right)^2$: Nonlinear strain, which accounts for the geometric effects resulting from large deformations.

Thus, the total strain is the sum of the linear strain and the nonlinear strain due to the large displacement. The nonlinear term is important for large deformation problems where the displacement and strain are no longer proportional (i.e., when the displacement field is nonlinear).

## 5.2 Deriving Element Tangential Stiffness

The strain energy $U$ is computed using the linear strain term, which corresponds to the elastic deformation in the material. It is expressed as:

$$U = \frac{1}{2}\int_0^L EA\varepsilon_L^2 \, dx = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx, \tag{45}$$

where $\varepsilon_L = \frac{du}{dx}$ is the linear strain and $A$ is the cross-sectional area. This term represents the elastic energy stored in the material due to deformation.

The nonlinear term contributes to the geometric stiffness via work equivalence. The axial force $N$ generates work through the nonlinear strain. This work is given by:

$$W_{\text{geo}} = \int_0^L N \cdot \varepsilon_{\text{NL}} \, dx = \int_0^L N\left(\frac{1}{2}\left(\frac{du}{dx}\right)^2\right) dx, \tag{46}$$

where $\varepsilon_{\text{NL}}$ represents the nonlinear strain term. This expression captures the geometric work resulting from the axial force and its effect on the system.

According to Section 4, for a nonlinear bar element,

$$U^e = \frac{1}{2}(\mathbf{u}^e)^T \underbrace{\left(\int_0^{L_e} EA\mathbf{B}^T\mathbf{B}dx\right)}_{\mathbf{k}_m^e} \mathbf{u}^e \tag{47}$$

$$W_{\text{geo}}^e = \frac{1}{2}(\mathbf{u}^e)^T \underbrace{\left(\int_0^{L_e} N\mathbf{B}^T\mathbf{B}dx\right)}_{\mathbf{k}_{\text{geo}}^e} \mathbf{u}^e \tag{48}$$

The **material stiffness matrix** $\mathbf{k}_m$ is derived from the linear strain energy, which accounts for the material properties (Young's modulus $E$, cross-sectional area $A$) and the strain-displacement relationship. For a linear element, the material stiffness matrix is:

$$\mathbf{k}_m^e = \int_0^{L_e} EA\mathbf{B}^T\mathbf{B}\, dx$$

where:

- $E$ is Young's modulus,
- $A$ is the cross-sectional area,
- $L_e$ is the length of the element,
- $\mathbf{B}$ is the strain-displacement matrix, see eq. 33.

Substituting $\mathbf{B}$ into the integral:

$$\mathbf{k}_m^e = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The **geometric stiffness matrix** $\mathbf{k}_{\text{geo}}^e$ accounts for the nonlinear effects due to large displacements, specifically the axial force. The axial force introduces additional stiffness, which can be captured by the nonlinear strain term.

For a nonlinear element, the geometric stiffness matrix is given by:

$$\mathbf{k}_{\text{geo}}^e = \int_0^{L_e} N\mathbf{B}^T\mathbf{B}\, dx$$

where:

- $N$ is the axial force,
- $\mathbf{B}$ is the strain-displacement matrix, which is the same as in the material stiffness case.

Substituting $\mathbf{B}$ into the integral:

$$\mathbf{k}_{\text{geo}}^e = \frac{N}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where:

- $N$ is the axial force (which depends on the displacement and the load),
- $L_e$ is the length of the element.

The total tangential stiffness matrix $\mathbf{k}_t$ is the sum of the material stiffness matrix and the geometric stiffness matrix:

$$\mathbf{k}_t^e = \mathbf{k}_m^e + \mathbf{k}_{\text{geo}}^e$$

Thus, combining both material and geometric contributions, we get the total stiffness matrix for the nonlinear element:

$$\mathbf{k}_t = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{N}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This expression incorporates both the material properties and the geometric effects, which is essential for accurately solving problems involving large deformations and axial forces.